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A NON-SYMMETRIC WEIGHTED DIVERGENCE AND KULLBACK-LEIBLER WEIGHTED DIVERGENCE MEASURE

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Abstract

Information theory literature is familiar with weighted information divergence measures and their bounds. In this study, a novel non-symmetric information weighted divergence measure will be taken into consideration. In terms of the Kullback- Leibler weighted divergence measure, the upper and lower bounds of non-symmetric information weighted divergence have been investigated.

Key words: "Csiszar's f-Divergence Measure, Kullback-Leibler weighted Divergence Measure, Information Inequalities etc."

1 INTRODUCTION

Let

$$\Gamma_n = \{P = (p_1, p_2, \ldots, p_n), p_i > 0, \quad \sum_{i=1}^{n} p_i = 1, n \geq 2$$

be the set of all weighted complete finite discrete probability distributions. "Csiszar (Csiszar, 1961) and (Csiszar, 1978)" presented a generalised measure of information using the f-divergence measure, which is given by

$$I_f(P; Q) = \sum_{i=1}^{n} q_i \, f\left(\frac{p_i}{q_i}\right)$$

And its weighted form is defined as given below by many authors as

$$I_f(P; Q; W) = \sum_{i=1}^{n} w_i \, q_i f\left(\frac{p_i}{q_i}\right), \qquad (1.1)$$

where $P, Q \in \Gamma_n$.

Here, along with the appropriate generating function f, we list existing weighted divergence measures that fall under the category of weighted Csiszar's f -divergence measures.

"Kullback-Leibler divergence measure (Kullback and Leibler, 1951)"

(i) If $f(\varepsilon) = -\log \varepsilon$, then the “Kullback & Leibler divergence measure” is provided by

$$I_f(P; Q) = D(Q; P) = \sum_{i=1}^{n} q_i \log\left(\frac{q_i}{p_i}\right) \quad (1.2)$$

(ii) If $f(\varepsilon) = \varepsilon \log \varepsilon$, then the “Kullback & Leibler divergence measure” is provided by

$$I_f(P; Q) = D(P; Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right) \quad (1.3)$$

& the weighted form of the “Kullback & Leibler divergence measure” is provided by

$$I_f(P; Q; W) = D(Q; P; W) = \sum_{i=1}^{n} w_i q_i \log\left(\frac{q_i}{p_i}\right) \quad (1.4)$$

$$I_f(P; Q; W) = D(P; Q; W) = \sum_{i=1}^{n} w_i p_i \log\left(\frac{p_i}{q_i}\right) \quad (1.5)$$

Information inequalities are a topic we cover in section 2 of the entire work. In section 3, a “new weighted non-symmetric information divergence measure” has been developed. Section 4 has investigated the bounds of the Kullback-Leibler divergence measure in weighted form for fresh information.

2 INEQUALITIES CONNECTED TO CSISZAR’S F-DIVERGENCE MEASURES

One of the outcomes of the theory stated in Taneja and Kumar (2004) is the following claim, which is along the same lines as “(Dragomir, 2001; Jain and Saraswat, 2013 and Jain and Saraswat, 2013)”

Proposition.2.1: - If $f : I \times (0, \infty) \subset \mathcal{R}_+ \times (0, \infty) \to \mathcal{R}$ the generating mapping is normalized, with f (1; 1) equal to 0, and it meets the following assumptions:

(i) f can be differentiated twice on (ℓ, L) , here $0 \le \ell \le 1 \le L \le \infty$,

(ii) there exist real constants λ, δ s.t. $0 < \lambda < \delta$ and

$$\lambda \le \varepsilon^{2-s} f''(\varepsilon; w) \le \delta \quad \forall \varepsilon \in (\ell, L), -\infty < s < \infty \quad (2.1)$$

Let P, $Q \in \Gamma_n$ be such that there exist ℓ, L with $0 < \ell \le \frac{p_i}{q_i} \le L < \infty$, for every i=1to n then we have

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$$\lambda\varphi_s(Ᵽ;Q;W) \leq I_f(Ᵽ;Q;W) \leq \delta\varphi_s(Ᵽ;Q;W) \qquad (2.2)$$

Proposition 2.1's instance s=0, s=1 results.

Proposition.2.2: - If $f: I \times (0,\infty) \subset \mathcal{R}_+ \times (0,\infty) \rightarrow \mathcal{R}$ the generating mapping is normalized, with f (1; 1) equal to 0, and it meets the following assumptions:

(i) f can be differentiated twice on (ℓ, L), here $0 \leq \ell \leq 1 \leq L \leq \infty$,

(ii) there exist real constants λ, δ s.t. $0 < \lambda < \delta$ and

$$\lambda \leq \varepsilon^2 f''(\varepsilon;w) \leq \delta \quad \forall\varepsilon \in (\ell, L), \qquad (2.3)$$
$$\lambda \leq \varepsilon f''(\varepsilon;w) \leq \delta \quad \forall\varepsilon \in (\ell, L), \qquad (2.4)$$

Let Ᵽ, Q $\in \Gamma_n$ be such that there exist ℓ, L with $0 < \ell \leq \frac{þ_i}{q_i} \leq L < \infty$, for every i=1to n then we have for s=0, s=1

$$\lambda D(Q;Ᵽ;W) \leq I_f(Ᵽ;Q;W) \leq \delta D(Q;Ᵽ;W) \qquad (2.5)$$

$$\lambda D(Ᵽ;Q;W) \leq I_f(Ᵽ;Q;W) \leq \delta D(Ᵽ;Q;W) \qquad (2.6)$$

The following conclusion can be stated in light of proposition 2.1.

3 WEIGHTED NON-SYMMETRIC DIVERGENCE MEASURE

Here we introduce the category of Csiszar's f -divergence measure, a novel information divergence measure.

Let's think about the function $f: I \times (0,\infty) \subset \mathcal{R}_+ \times (0,\infty) \rightarrow \mathcal{R}$

$$f(\varepsilon,w) = \frac{w(\varepsilon^2-1)^2}{\varepsilon},\quad f'(\varepsilon;w) = \frac{w[3\varepsilon^4-2\varepsilon^2+1)]}{\varepsilon^2},\quad f''(\varepsilon;w) = \frac{w[6\varepsilon^4+2]}{\varepsilon^3} > 0, \text{for all } \varepsilon > 0, w>0 \qquad (3.1)$$

So $f(\varepsilon,w)$ is convex from (3.1) & f(1,1)=0 (i.e., normalized)

Function (3.1)'s associated f - divergence measure in weighted form is given by

$$I_f(Ᵽ;Q;W) = \sum_{i=1}^{n} w_i\, q_i f\left(\frac{þ_i}{q_i}\right) = \sum_{i=1}^{n} w_i \frac{(þ_i^2-q_i^2)^2}{q_i^2 þ_i}$$

$$= 4\sum_{i=1}^{n} w_i \left(1/\frac{2þ_i q_i}{þ_i+q_i}\right)\frac{þ_i+q_i}{2}\frac{(þ_i-q_i)^2}{q_i} = N(Ᵽ;Q;W) \qquad (3.2)$$

Where "$N(Ᵽ;Q;W)$" can be combination of "weighted Harmonic, Arithmetic & Chi- square divergence measures"

Further $f(1,1) = 0$, so $N(Ᵽ; Ᵽ; W)$ = 0 & from convexity of $f(\varepsilon, w)$ the measure $N(Ᵽ; Q; W)$ is non negative.

Using the measure $N(Ᵽ; Q; W)$ given in equation, we present specific results of the proposition (2.1) in the section 4 that follow (3.2)

4. BOUNDS USING THE KULLBACK-LEIBLER DIVERGENCE MEASURE

Result 4.1: - Let Ᵽ, Q $\in \Gamma_n$ and s = 0. Let there exists ℓ, L such that ℓ < L &

$$0 < \ell \le \frac{þ_i}{q_i} \le L < \infty, \text{ for every i=1to } n.$$

(i) If $\ell \in \left(0, \frac{1}{\sqrt{3}}\right)$ then

$$\frac{8\sqrt{3}}{3} wD(Q; Ᵽ; W) \approx 4.61 wD(Q; Ᵽ; W)$$

$$\le N(Ᵽ; Q; W) \le \max\left(\frac{w(6\ell^4+2)}{\ell}, \frac{w(6L^4+2)}{L}\right) D(Q; Ᵽ; W) \quad (4.1)$$

(ii) If $\ell \in \left(\frac{1}{\sqrt{3}}, \infty\right)$ then

$$\frac{w(6L^4+2)}{L} D(Q; Ᵽ; W) \le N(Ᵽ; Q; W) \le \frac{w(6\ell^4+2)}{\ell} D(Q; Ᵽ; W) \quad (4.2)$$

Proof: - From (3.1), (3.2) & (4.1), we have

$k(\varepsilon, w) = \varepsilon^2 f''(\varepsilon; w) = \frac{w[6\varepsilon^4+2]}{\varepsilon}$ for all $\varepsilon > 0$, w>0 we have

$k'(\varepsilon; w) = w\left(18\varepsilon^2 - \frac{2}{\varepsilon^2}\right)$

$k'(\varepsilon; w) = 0$ gives $\varepsilon_0 = \frac{1}{\sqrt{3}} \approx .58$

$k''(\varepsilon; w) = w\left(36\varepsilon + \frac{4}{\varepsilon^3}\right)$ & $k''(.58; w) = 49.3w > 0$, for w > 0

this demonstrates that the minimum value of the function $k(\varepsilon, w)$ occurs at $\varepsilon_0 = .58$ & min of $k(\varepsilon_0, w) = \lambda$, we have two cases:

(i) $0 < \ell \le \frac{1}{\sqrt{3}}$ then

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$$\lambda = \inf_{\varepsilon\in[\ell,L]} k(\varepsilon; w) = k(\varepsilon_0, w) = \frac{8\sqrt{3}w}{3} \approx 4.61w$$

$$\delta = \sup_{\varepsilon\in[\ell,L]} k(\varepsilon; w) = \max\left(\frac{w(6\ell^4+2)}{\ell}, \frac{w(6L^4+2)}{L}\right) \quad (4.3)$$

(ii) $\frac{1}{\sqrt{3}} < \ell < \infty$,then

$$\lambda = \inf_{\varepsilon\in[\ell,L]} k(\varepsilon; w) = \frac{w(6\ell^4+2)}{\ell}, \delta = \sup_{\varepsilon\in[\ell,L]} k(\varepsilon; w) = \frac{w(6L^4+2)}{L} \quad (4.4)$$

Equations (2.5) & (2.6) of proposition (2.2) produce the results (4.1) & (4.2) using equations (3.2), (4.3), and (4.4).

Result 4.2: - Let Ᵽ, Ọ $\in \Gamma_n$ and s = 1 Let there exists ℓ, L such that ℓ < L &

$$0 < \ell \le \frac{p_i}{q_i} \le L < \infty, \text{ for every i=1to } n.$$

(i) $0 < \ell \le 0.76$, then

$$4\sqrt{3}\, wD(Ᵽ; Ọ; W) \le N(Ᵽ; Ọ; W) \le \max\left(\frac{w(6\ell^4+2)}{\ell}, \frac{w(6L^4+2)}{L}\right) D(Ᵽ; Ọ; W) \quad (4.5)$$

(ii) $0.76 < \ell < \infty$, then

$$\frac{w(6\ell^4+2)}{\ell} D(Ᵽ; Ọ; W) \le N(Ᵽ; Ọ; W) \le \frac{w(6L^4+2)}{L} D(Ᵽ; Ọ; W) \quad (4.6)$$

Proof: - From (3.1), (3.2) & (4.2), we have

$k(\varepsilon, w) = \varepsilon f''(\varepsilon; w) = \frac{w[6\varepsilon^4+2]}{\varepsilon^2}$ for all $\varepsilon > 0, w>0$ we have

$k'(\varepsilon; w) = w\left(12\varepsilon - \frac{4}{\varepsilon^3}\right)$

$k'(\varepsilon; w) = 0$ gives $\varepsilon_0 = (\frac{1}{3})^{\frac{1}{4}} \approx 0.76$

$k''(\varepsilon; w) = w\left(12 + \frac{12}{\varepsilon^4}\right)$ & $k''(0.76; w) > 0$, for $w > 0$

this demonstrates that the minimum value of the function k(ε,w) occurs at $\varepsilon_0 = (\frac{1}{3})^{\frac{1}{4}} \approx 0.76$

we have two cases:

(i) $0 < \ell \leq 0.76$, then

$$\lambda = \inf_{\varepsilon\in[\ell,L]} k(\varepsilon; w) = k(\varepsilon_0, w) = 4\sqrt{3}\,w \qquad (4.7)$$

$$\delta = \sup_{\varepsilon\in[\ell,L]} k(\varepsilon; w) = \max\left(\frac{w(6\ell^4+2)}{\ell}, \frac{w(6L^4+2)}{L}\right) \qquad (4.8)$$

(ii) $0.76 < \ell < \infty$, then

$$\lambda = \inf_{\varepsilon\in[\ell,L]} k(\varepsilon; w) = \frac{w(6\ell^4+2)}{\ell},\ \delta = \sup_{\varepsilon\in[\ell,L]} k(\varepsilon; w) = \frac{w(6L^4+2)}{L} \qquad (4.9)$$

Equations (2.6) and (2.7) of proposition (2.2) generate the outcomes (4.5) & (4.6) using equations (3.2), (4.7), (4.8), & (4.9).

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